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# Analysis of a degenerate travelling wave instability

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**Abstract.** A bifurcation of codimension three where two travelling wave instabilities merge is investigated. For this case the amplitude equation is derived in detail for a general system using a multiple scale expansion. It contains non-local contributions. Complete solutions of this equation and their asymptotic stability can be calculated analytically.

## 1. Introduction

In recent years much work has been done to understand pattern formation in spatially extended systems [1–3]. One typically finds such codimension-one bifurcations as soft-mode, hard-mode and travelling wave instabilities in a vast class of systems ranging from fluids and lasers to magnets and chemical systems. A powerful method in dealing with these problems, especially in spatially low-dimensional systems, is the description in terms of amplitude equations. They provide an apt tool to analyse patterns above instabilities of a trivial solution and lead to Ginzburg–Landau equations for the amplitude functions.

While knowledge of codimension-one instabilities is great, bifurcations of higher codimension in spatially extended systems are less well understood. These usually appear where coefficients of the Ginzburg–Landau equation become zero or where different instabilities of the trivial solution occur simultaneously. One bifurcation of the first kind is discussed in [4] where the cubic coefficient of the amplitude equation of a soft-mode instability vanishes. One prominent example of the latter case is the Turing–Hopf bifurcation, where a soft-mode and a hard-mode instability occur simultaneously. It has been shown that the dynamics above threshold can be described by two coupled Ginzburg–Landau equations [5,6]. The solutions of this complicated system are often studied numerically.

An analogous codimension-two case is the intersection of two travelling wave lines. At this point the spectrum of the linearized system shows two critical wavenumbers $k_{c,1}, k_{c,2} \neq 0$ with non-vanishing frequencies. We have found such a situation in a one-dimensional damped ferromagnet being driven by a propagating magnetic wave. The details of the magnetic system are given in appendix A. Any variation of the physical parameters describing the system will generally change the difference between the two wavenumbers involved, which indeed can even become zero. In such a case which arises quite generically the two travelling wave instabilities merge in a codimension-three bifurcation. The situation can be viewed either as a merging of two travelling wave instabilities or as a degeneration of a travelling wave instability giving rise to two bifurcation lines. Of course this codimension-three bifurcation is more difficult to detect than the more usual soft-mode, hard-mode or travelling wave instabilities.

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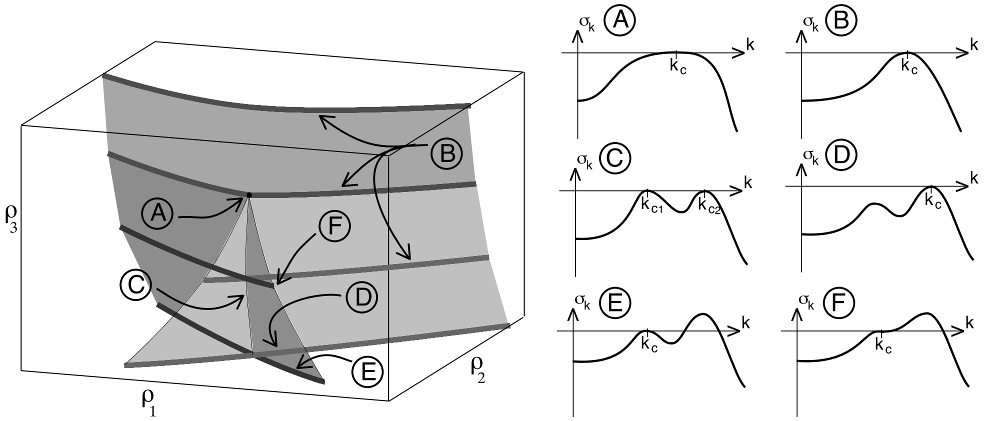


**Figure 1.** Bifurcation diagram and real part of spectra. $\rho_1$, $\rho_2$ and $\rho_3$ are parameters of a general system. Thick lines correspond to bifurcation lines in planes $\rho_3 = \text{const}$.

## 2. Bifurcation diagram and spectra

In this paragraph we consider a general spatially one-dimensional system and specify the degeneracy that is required to obtain the case described above. Near the travelling wave instability the spectrum of the linearized problem is expanded around the critical wavenumber $k_c$.

$$\lambda_k = \sigma_k + \mathrm{i}\omega_k = \lambda_{k_c} + (k - k_c)\partial_k\lambda_k|_{k_c} + (k - k_c)^2/2\partial_k^2\lambda_k|_{k_c} + \cdots . \qquad (1)$$

In the non-degenerate case the maximum of the real part $\sigma_k$ is quadratic and higher-order terms in this expansion need not be considered. Now let the degeneracy be such that the real part of the quadratic coefficient $\partial_k^2\lambda_k|_{k_c}$ vanishes, resulting in a fourth-order maximum where $\sigma_k \propto (k - k_c)^4$.

The structure of the corresponding bifurcation surface, as well as the real part of the spectrum of the linear system, are shown schematically in figure 1. Point (A) indicates the degenerate travelling wave bifurcation in the parameter space of the general system.

In the vicinity of point (A) the quadratic coefficient of the Taylor expansion changes its sign. If it is negative, there is only one maximum in $\sigma_k$ corresponding to the single travelling wave line in the upper part of the diagram. The real part of the spectrum for this case is sketched in (B). For positive coefficients $\partial_k^2\lambda_k|_{k_c}$ the quartic maxima at point (A) breaks up into two quadratic maxima hence giving rise to two bifurcation lines for waves with different critical wavenumbers $k_{c,1}$ and $k_{c,2}$. The intersection of these two lines marks a point of codimension two (C) where both instabilities occur simultaneously, similar to the Turing–Hopf† point. Usually the two maxima in $\sigma_k$ will be of different height. Then the more elevated one (D) causes the primal instability in the system. The bifurcation connected with the lower maximum is less interesting (E), since the trivial solution is already unstable with respect to wavenumbers near the other maximum. A bifurcation line ends, when the lower maximum meets the minimum in $\sigma_k$ and becomes a saddle point (F). Note that in the vicinity of the degenerate bifurcation the real part of the spectrum locally is a polynomial in $(k - k_c)$ of order four.

While in the meantime, pattern formation near a single travelling wave bifurcation line has become a textbook example, the derivation of an amplitude equation near the degenerate

† In the magnetic model described in appendix A it indeed becomes a Turing–Hopf bifurcation provided the inversion symmetry is restored, which happens for spatially homogeneous driving fields.

travelling wave bifurcation turns out to be much less trivial. The reason for this is that the frequencies $\omega_k$ cannot be eliminated by a simple transformation, which could be achieved in the non-degenerate case by changing to a comoving frame of reference. We have been led to apply a non-standard technique in order to solve this problem.

## 3. General system and notation

In what follows we analyse the degenerate travelling wave instability for a quite general spatially one-dimensional system with broken inversion symmetry†. The system is by assumption autonomous and invariant to translations. Its spatial size is $L$ and periodic boundary conditions are used. The equation of motion for the vector field $\underline{\Phi}$, which measures the deviation from some homogeneous equilibrium state, reads

$$\partial_t \underline{\Phi} = \underline{\mathcal{L}}[\underline{\Phi}] + \underline{\mathcal{N}}[\underline{\Phi}] \qquad \underline{\Phi}(x,t) : [0, L] \times \mathbb{R} \Rightarrow \mathbb{R}^n. \tag{2}$$

The linear and nonlinear operators $\underline{\mathcal{L}}$ and $\underline{\mathcal{N}}$ depend on the set of external parameters $\underline{r}$ and may contain derivatives of arbitrary order with respect to $x$. They are written as

$$\begin{aligned} &\underline{\mathcal{L}}[\underline{\Phi}] = \sum_\alpha \underline{\underline{B}}_\alpha(\underline{r})\partial_x^\alpha \underline{\Phi} \qquad \underline{\mathcal{N}}[\underline{\Phi}] = \underline{\mathcal{N}}_2[\underline{\Phi}] + \underline{\mathcal{N}}_3[\underline{\Phi}] + \cdots \qquad \underline{r} = (\rho_1, \rho_2, \rho_3, \ldots) \\ &\underline{\mathcal{N}}_2[\underline{\Phi}] = \sum_{\alpha,\beta} \underline{C}_{\alpha,\beta}(\underline{r})\{\partial_x^\alpha \underline{\Phi}, \partial_x^\beta \underline{\Phi}\} \qquad \underline{\mathcal{N}}_3[\underline{\Phi}] = \sum_{\alpha,\beta,\gamma} \underline{D}_{\alpha,\beta,\gamma}(\underline{r})\{\partial_x^\alpha \underline{\Phi}, \partial_x^\beta \underline{\Phi}, \partial_x^\gamma \underline{\Phi}\}. \end{aligned} \tag{3}$$

In these formulae the $\underline{\underline{B}}_\alpha$ are matrices, whereas the tensor functions $\underline{C}$ and $\underline{D}$ can be chosen to be symmetric in their arguments, i.e. $\underline{C}_{\alpha,\beta}\{\underline{u}, \underline{v}\} = \underline{C}_{\beta,\alpha}\{\underline{v}, \underline{u}\}$, $\underline{D}_{\alpha,\beta,\gamma}\{\underline{u}, \underline{v}, \underline{w}\} = \underline{D}_{\alpha,\gamma,\beta}\{\underline{u}, \underline{w}, \underline{v}\}$, etc. The eigenvalue problem of $\underline{\mathcal{L}}$ is solved in terms of Fourier modes:

$$\begin{aligned} &\lambda_k^{(\nu)} \underline{\varphi}_k^{(\nu)} = \underline{\mathcal{L}}[\underline{\varphi}_k^{(\nu)}] \qquad \underline{\varphi}_k^{(\nu)} = \underline{u}_k^{(\nu)} \mathrm{e}^{\mathrm{i}kx} \qquad \lambda_k^{(\nu)} = \sigma_k^{(\nu)} + \mathrm{i}\omega_k^{(\nu)} \\ &\Longrightarrow \lambda_k^{(\nu)} \underline{u}_k^{(\nu)} = \sum_\alpha \underline{\underline{B}}_\alpha(\underline{r})(\mathrm{i}k)^\alpha \underline{u}_k^{(\nu)} =: \underline{\underline{L}}_k \underline{u}_k^{(\nu)}. \end{aligned} \tag{4}$$

The eigenvalues $\lambda_k^{(\nu)}$ of the different branches are enumerated by the index $\nu$. In this spectrum there exists by assumption only one unstable mode labelled by $\nu_c$, so that along the heavy bifurcation lines in the corresponding region of parameter space (cf figure 1) $\sigma_k^{(\nu_c)}$ obeys $\sigma_k^{(\nu_c)}|_{k_c,\underline{r}_c} = 0$, $\partial_k \sigma_k^{(\nu_c)}|_{k_c,\underline{r}_c} = 0$. Now, as a consequence of the discussion in the previous section, at point (A) an additional degeneration occurs, namely

$$\partial_k^2 \sigma_k^{(\nu_c)}|_{k_c,\underline{r}_c} = 0 \qquad \partial_k^3 \sigma_k^{(\nu_c)}|_{k_c,\underline{r}_c} = 0. \tag{5}$$

Near $k_c$, the imaginary part of the spectrum $\omega_k^{(\nu_c)}$ is an arbitrary function of $k$. This means that the frequencies in the equations of motion cannot be eliminated by a transformation to some comoving frame of reference. As sketched in the picture above, we consider only non-vanishing critical wavenumbers.

As our basic equations of motion are translational invariant they may be transformed into equations for corresponding Fourier amplitudes via the decomposition

$$\underline{\Phi}(x,t) = \frac{1}{\sqrt{L}} \sum_{k,\nu} \Phi_k^{(\nu)}(t) \underline{u}_k^{(\nu)} \mathrm{e}^{\mathrm{i}kx} \tag{6}$$

† An inversion symmetric system would generally produce not one but two counterpropagating waves. Excluding further degeneracies one finds non-local amplitude equations for this case, as was pointed out by Knobloch and de Luca [7].

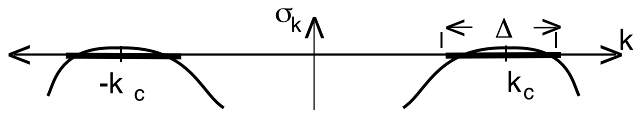


**Figure 2.** Interval of active modes (bold segments).

which gives

$$\partial_t \Phi_k^{(\nu)} = \lambda_k^{(\nu)} \Phi_k^{(\nu)} + \frac{1}{\sqrt{L}} \sum_{k_1,\nu_1,k_2,\nu_2} \delta_{k,k_1+k_2} \mathrm{R}_{kk_1k_2}^{\nu\nu_1\nu_2} \Phi_{k_1}^{(\nu_1)} \Phi_{k_2}^{(\nu_2)} + \frac{1}{L} \sum_{k_1,\nu_1,k_2,\nu_2,k_3,\nu_3} \delta_{k,k_1+k_2+k_3} \mathrm{S}_{kk_1k_2k_3}^{\nu\nu_1\nu_2\nu_3} \Phi_{k_1}^{(\nu_1)} \Phi_{k_2}^{(\nu_2)} \Phi_{k_3}^{(\nu_3)} + \cdots. \tag{7}$$

The contributions of the nonlinear operator $\underline{\mathcal{N}}$ contain the left eigenvectors $\underline{v}_k^{(\nu)}$ of $\underline{\underline{L}}_k$:

$$\begin{aligned} \mathrm{R}_{k,k_1k_2}^{\nu\nu_1\nu_2} &= \sum_{\alpha,\beta} (\mathrm{i}k_1)^\alpha (\mathrm{i}k_2)^\beta (\underline{v}_k^{(\nu)} \cdot \underline{C}_{\alpha,\beta}\{\underline{u}_{k_1}^{(\nu_1)}, \underline{u}_{k_2}^{(\nu_2)}\}) \\ \mathrm{S}_{kk_1k_2k_3}^{\nu\nu_1\nu_2\nu_3} &= \sum_{\alpha,\beta,\gamma} (\mathrm{i}k_1)^\alpha (\mathrm{i}k_2)^\beta (\mathrm{i}k_3)^\gamma (\underline{v}_k^{(\nu)} \cdot \underline{D}_{\alpha,\beta,\gamma}\{\underline{u}_{k_1}^{(\nu_1)}, \underline{u}_{k_2}^{(\nu_2)}, \underline{u}_{k_3}^{(\nu_3)}\}). \end{aligned} \tag{8}$$

Since we start from a system with finite, though very large length $L$, $k$ is a discrete index. The limit $L \to \infty$ will be performed after the derivation of amplitude equations has been completed. This is a physicist's approach to the problem, where emphasis is not on mathematical rigour†.

## 4. Weakly nonlinear analysis

Equation (7) describes the dynamics of all Fourier modes in all branches, whether they are damped or not. However, only a small fraction of them, namely the linearly unstable or 'active' modes [2] respectively, turn out to be constitutive or relevant for pattern formation. The linearly stable modes are damped and, as a consequence, enslaved by the active ones [8]. They will be eliminated adiabatically by applying a reduction scheme in analogy to the centre manifold reductions for low-dimensional systems.

Figure 2 makes it obvious how to choose the active modes. They are determined by the spectrum of the linear operator $\mathcal{L}$ and lie symmetrically with respect to the origin in the two intervals $\mathrm{I}_\pm = (\pm k_c - \Delta/2, \pm k_c + \Delta/2)$. Their width is given by $\Delta \propto \sqrt[4]{|\sigma_{k_c}^{(\nu_c)}|}$.

The elimination of the passive modes, which consist of all modes $\mu \neq \nu_c$ and all modes outside the interval $\mathrm{I} = \mathrm{I}_+ \cup \mathrm{I}_-$ for $\nu = \nu_c$, respectively, is achieved by applying the Taylor expansion

$$q \notin \mathrm{I} \quad \text{or} \quad \mu \neq \nu_c: \qquad \Phi_q^{(\mu)} = \frac{1}{\sqrt{L}} \sum_{k_1,k_2 \in \mathrm{I}} \alpha_{q,k_1,k_2}^{\mu,\nu_c,\nu_c} \Phi_{k_1}^{(\nu_c)} \Phi_{k_2}^{(\nu_c)} + \mathrm{O}(\Phi_k^{(\nu_c)^3}). \tag{9}$$

The coefficients $\alpha_{q,k_1,k_2}^{\mu,\nu_c,\nu_c}$ which are unknown initially result by inserting this 'ansatz' into (7) and comparing terms up to second order in the amplitudes $\Phi_k^{(\nu_c)}$:

$$\alpha_{q,k_1,k_2}^{\mu,\nu_c,\nu_c} = -\frac{\delta_{k,k_1+k_2} \mathrm{R}_{q\,k_1\,k_2}^{\mu\nu_c\nu_c}}{\lambda_q^{(\mu)} - \lambda_{k_1}^{(\nu_c)} - \lambda_{k_2}^{(\nu_c)}}. \tag{10}$$

† It seems that the limit $L \to \infty$ may be performed as usual by replacing $\frac{1}{L}\sum_k \to \frac{1}{(2\pi)^{1/2}} \int \mathrm{d}k$ and Kronecker by Dirac deltas. $\delta_{k,k'} \to \delta(k - k')$. This is at least true for the discussions in section 4 and appendix A.

The dynamics of the active modes can now be written down explicitly up to third order. For our purpose higher-order terms can be neglected.

$$\partial_t \Phi_k^{(\nu_c)} = \lambda_k^{(\nu_c)} \Phi_k^{(\nu_c)} + \frac{1}{L} \sum_{k_j \in \mathrm{I}} \delta_{k,k_1+k_2+k_3} \mathrm{T}_{kk_1k_2k_3} \Phi_{k_1}^{(\nu_c)} \Phi_{k_2}^{(\nu_c)} (\Phi_{k_3}^{(\nu_c)})^* + \mathrm{O}(\Phi_k^4). \tag{11}$$

$$\mathrm{T}_{kk_1k_2k_3} = \mathrm{S}_{kk_1k_2k_3}^{\nu_c\nu_c\nu_c\nu_c} - 2 \sum_{q \notin \mathrm{I},\mu} \mathrm{R}_{kk_1q}^{\nu_c\nu_c\mu} \frac{\delta_{k,k_1+q}}{\lambda_q^{(\mu)} - (\lambda_{k_2}^{(\nu_c)} + \lambda_{k_3}^{(\nu_c)})} \mathrm{R}_{qk_2k_3}^{\mu\nu_c\nu_c}. \tag{12}$$

Note that because $\Delta$ is much less than $k_c/3$ quadratic terms do not appear in equation (11) because of the conservation of momentum. To study the system of equations (11) near threshold where $\underline{r} - \underline{r}_c = \mathrm{O}(\varepsilon^2)$, the method of multiple scales is applied. Expanding the spectrum in a Taylor series

$$\lambda_k^{(\nu_c)} = \mathrm{i}(\omega_{k_c,\underline{r}_c}^{(\nu_c)} + (k-k_c)\partial_k \omega_k^{(\nu_c)}|_{k_c,\underline{r}_c} + \tfrac{1}{2}(k-k_c)^2 \partial_k^2 \omega_k^{(\nu_c)}|_{k_c,\underline{r}_c} + \tfrac{1}{6}(k-k_c)^3 \partial_k^3 \omega_k^{(\nu_c)}|_{k_c,\underline{r}_c})$$
$$+ \frac{1}{4!}(k-k_c)^4 \partial_k^4 \lambda_k^{(\nu_c)}|_{k_c,\underline{r}_c} + (\underline{r} - \underline{r}_c)\partial_{\underline{r}} \lambda_k^{(\nu_c)}|_{k_c,\underline{r}_c} + \cdots \tag{13}$$

leads to supposing that $\mathrm{O}((k-k_c)^4) = \mathrm{O}(\underline{r} - \underline{r}_c)$. Therefore the width $\Delta$ of the interval is $\mathrm{O}(\sqrt{\varepsilon})$. Consequently the possible timescales of the oscillations of the Fourier modes are $t, \sqrt{\varepsilon}t, \varepsilon t$, etc. Since the system is invariant with respect to translations we can avoid dealing with the fastest oscillations by choosing an appropriate frame of reference. The transformation†:

$$\Psi_k = \Phi_k^{(\nu_c)} \mathrm{e}^{\mathrm{i}(\omega_c + (k-k_c)\partial_k \omega_k|_{k_c})t} \tag{14}$$

eliminates the first two terms in the RHS of (13), but leaves equation (11) otherwise unchanged. A systematic treatment will be accomplished by putting

$$\begin{aligned} &\underline{r} = \underline{r}_c + \varepsilon^2 \underline{r}_2 \qquad (k-k_c) = \mathrm{O}(\sqrt{\varepsilon}) \qquad t_n = \varepsilon^n t \\ &\Psi_k = \varepsilon \psi_k^{(1)} + \varepsilon\sqrt{\varepsilon}\psi_k^{(3/2)} + \varepsilon^2 \psi_k^{(2)} + \varepsilon^2\sqrt{\varepsilon}\psi_k^{(5/2)} + \cdots \\ &\partial_t = \partial_{t_0} + \sqrt{\varepsilon}\partial_{t_{1/2}} + \cdots \qquad \psi_k^{(i)} = \psi_k^{(i)}(t_0, t_{1/2}, t_1, \ldots). \end{aligned} \tag{15}$$

Insertion of (15) into (11) and requiring that the resulting system of equations is satisfied in all orders of $\varepsilon$ yields the following hierarchy of linear inhomogeneous differential equations for the $\psi_k^{(i)}$:

$$\begin{aligned} &[\varepsilon^{m/2}] : \sum_{n=0}^{m-2} \partial_{t_{n/2}} \psi_k^{(\frac{m-n}{2})} = \mathrm{i} \sum_{n=0}^{m-2} \Omega_k^{(n/2)} \psi_k^{(\frac{m-n}{2})} \qquad m \in \{2,3,4,5\} \\ &[\varepsilon^3] : \sum_{n=0}^{4} \partial_{t_{n/2}} \psi_k^{(3-\frac{n}{2})} = \mathrm{i} \sum_{n=0}^{4} \Omega_k^{(n/2)} \psi_k^{(3-\frac{n}{2})} + \mu \psi_k^{(1)} + \alpha_k \psi_k^{(1)} \\ &\qquad + \frac{\gamma}{L} \sum_{k_1,k_2,k_3 \in \mathrm{I}_+} \delta_{k,k_1+k_2-k_3} \psi_{k_1}^{(1)} \psi_{k_2}^{(1)} \psi_{k_3}^{(1)*}. \end{aligned} \tag{16}$$

In writing equations (16) the following abbreviations have been used.

$$\begin{aligned} &\Omega_k^{(0)} = \Omega_k^{(1/2)} = 0 \qquad \Omega_k^{(n/2)} = \frac{1}{n!}\partial_k^n \omega_k^{(\nu_c)}|_{k_c,\underline{r}_c} \hat{k}^n \qquad n \in \{2,3\} \qquad \hat{k} = \frac{k-k_c}{\sqrt{\varepsilon}} \\ &\mu = \underline{r}_2 \cdot \partial_{\underline{r}} \lambda_k^{(\nu_c)}|_{k_c,\underline{r}_c} \qquad \alpha_k = \frac{1}{4!}\partial_k^4 \lambda_k^{(\nu_c)}|_{k_c,\underline{r}_c} \hat{k}^4 \qquad \gamma = 2T_{kkk-k} + T_{k-kkk}|_{k_c,\underline{r}_c}. \end{aligned} \tag{17}$$

We now have to check under what conditions the hierarchy (16) has bounded solutions for any $m$. Up to order $\varepsilon^{5/2}$ this is fairly simple. For $m = 2$ equation (16) reads $\partial_{t_0} \psi_k^{(1)} = 0$, which leads to the conclusion that $\psi_k^{(1)} = \psi_k^{(1)}(t_{1/2}, t_1, \ldots)$ is not dependent on time $t_0$.

† We drop the index $\nu_c$ for brevity. It was the only upper index in (11).

Using this result the equation for $m = 3$ is a linear inhomogeneous differential equation for $\psi_k^{(3/2)}$ in the variable $t_0$, where the inhomogeneity, i.e. $\partial_{t_{1/2}}\psi_k^{(1)}$ is independent of $t_0$. Then, in order to avoid secular solutions the condition $\partial_{t_{1/2}}\psi_k^{(1)} = 0$ has to be fulfilled. This gives $\psi_k^{(1)} = \psi_k^{(1)}(t_1, \ldots)$ and also $\psi_k^{(3/2)} = \psi_k^{(3/2)}(t_{1/2}, t_1, \ldots)$.

This argument is applied twice for the variables $t_0$ and $t_{1/2}$ in the case $m = 4$ and produces the secular condition $\partial_{t_1}\psi_k^{(1)} = \mathrm{i}\Omega_k^{(1)}\psi_k^{(1)}$. Hence we get $\psi_k^{(1)} = \eta_k^{(1)}(t_{3/2}, t_2, \ldots)\mathrm{e}^{\mathrm{i}\Omega_k^{(1)}t_1}$ as well as the additional result that $\psi_k^{(3/2)}$ and $\psi_k^{(2)}$ do not depend on $t_0$ and $t_{1/2}$ respectively. Repeating this type of argument over and over again for $m = 5$ one obtains solutions up to order $\varepsilon^{5/2}$:

$$\begin{aligned} &\psi_k^{(1)} = \xi_k^{(1)}(t_2, \ldots)\mathrm{e}^{\mathrm{i}\Omega_k^{(1)}t_1}\mathrm{e}^{\mathrm{i}\Omega_k^{(3/2)}t_{3/2}} \qquad \psi_k^{(3/2)} = \eta_k^{(3/2)}(t_{3/2}, t_2, \ldots)\mathrm{e}^{\mathrm{i}\Omega_k^{(1)}t_1} \\ &\psi_k^{(2)} = \psi_k^{(2)}(t_1, t_{3/2}, t_2, \ldots) \qquad \psi_k^{(5/2)} = \psi_k^{(5/2)}(t_{1/2}, t_1, t_{3/2}, t_2, \ldots). \end{aligned} \tag{18}$$

We have not yet arrived at equations for the amplitudes which ensure saturation of the unstable modes $\psi_k$. This is achieved in $\mathrm{O}(\varepsilon^3)$. The secular condition on the timescale $t_1$ reads

$$\begin{aligned} (\partial_{t_1} - \mathrm{i}\Omega_k^{(1)})\psi_k^{(2)} = &-(\partial_{t_{3/2}} - \mathrm{i}\Omega_k^{(3/2)})\eta_k^{(3/2)}\mathrm{e}^{\mathrm{i}\Omega_k^{(1)}t_1} - (\partial_{t_2} - \mu - \alpha_k)\xi_k^{(1)}\mathrm{e}^{\mathrm{i}\Omega_k^{(1)}t_1 + \mathrm{i}\Omega_k^{(3/2)}t_{3/2}} \\ &+\frac{\gamma}{L}\sum_{k_1,k_2,k_3\in\mathrm{I}_+}\delta_{k,k_1+k_2-k_3}\xi_{k_1}^{(1)}\xi_{k_2}^{(1)}\xi_{k_3}^{(1)*}\mathrm{e}^{\mathrm{i}(\Omega_{k_1}^{(1)}+\Omega_{k_2}^{(1)}-\Omega_{k_3}^{(1)})t_1+\mathrm{i}(\Omega_{k_1}^{(3/2)}+\Omega_{k_2}^{(3/2)}-\Omega_{k_3}^{(3/2)})t_{3/2}}. \end{aligned} \tag{19}$$

In order to avoid resonances with frequency $\Omega_k^{(1)}$ all terms which obey $k^2 = k_1^2 + k_2^2 - k_3^2$ —remember that $\Omega_k^{(1)}$ is quadratic in $k - k_c$—have to be extracted from the triple sum on the RHS of equation (19) and after combining them with the remaining terms on the RHS put equal to zero. This yields

$$\begin{aligned} (\partial_{t_{3/2}} - \mathrm{i}\Omega_k^{(3/2)})\eta_k^{(3/2)} = &-(\partial_{t_2} - \mu - \alpha_k)\xi_k^{(1)}\mathrm{e}^{\mathrm{i}\Omega_k^{(3/2)}t_{3/2}} \\ &+\frac{\gamma}{L}\sum_{k_1,k_2,k_3\in\mathrm{I}_+}\delta_{k,k_1+k_2-k_3}\delta_{k^2,k_1^2+k_2^2-k_3^2}\xi_{k_1}^{(1)}\xi_{k_2}^{(1)}\xi_{k_3}^{(1)*}\mathrm{e}^{\mathrm{i}(\Omega_{k_1}^{(3/2)}+\Omega_{k_2}^{(3/2)}-\Omega_{k_3}^{(3/2)})t_{3/2}}. \end{aligned} \tag{20}$$

Now the conditions $k = k_1 + k_2 - k_3$ and $k^2 = k_1^2 + k_2^2 - k_3^2$ can only be fulfilled by choosing $k = k_1, k_2 = k_3$ or $k = k_2, k_1 = k_3$. Therefore only pairs $(k, -k)$ contribute to the dynamics of $\psi_k$ above threshold. The same reasoning—there ought to be no terms proportional to $\mathrm{e}^{\mathrm{i}\Omega_k^{(3/2)}t_{3/2}}$ on the RHS of equation (20)—leads to us requiring

$$\partial_{t_2}\psi_k^{(1)} = \mu\psi_k^{(1)} + \alpha_k\psi_k^{(1)} + 2\gamma\left(\frac{1}{L}\sum_{k'\in\mathrm{I}_+}|\psi_{k'}^{(1)}|^2\right)\psi_k^{(1)} \tag{21}$$

where we went back from $\xi_k^{(1)}$ to $\psi_k^{(1)}$ using equation (18). We remark that the fact that the sum on the RHS of equation (20) is restricted to pairs entails that it altogether contains only resonant contributions with respect to $t_{3/2}$. Equation (21) describes the saturation of the unstable modes which was sought for. Unlike the corresponding saturation term in the amplitude equation for the non-degenerate travelling wave instability, it is non-local in ordinary space. The source of this non-locality is very different from the one in the paper of Knobloch and de Luca [7], where the non-local character stems from the two different local coordinate systems for the counterpropagating waves. In a frame of reference moving with one of them, the other one shows up as an averaged quantity.

Due to the lack of inversion symmetry in the system analysed here there is only one wave offering a definite pertinent frame of reference. It is the oscillation of the Fourier modes with frequencies $\Omega_k^{(1)} \propto (k - k_c)^2$ on timescales which are smaller than the scale $t_2$ on which saturation occurs, which generates the non-locality in our case. The secularity condition being caused by these oscillations yields the constraint $k^2 = k_1^2 + k_2^2 - k_3^2$. This condition together with $k = k_1 + k_2 - k_3$, i.e. the conservation of momentum, reduces the triple sum of nonlinear terms to a simple sum, thereby generating the final non-local result equation (21).

It is important to observe that it is the imaginary part of the spectrum which produces these oscillations on the timescale $t_1$ above threshold. They cannot be eliminated from the problem but, as has been shown, may be taken into account properly by applying an appropriate version of the method of multiple scales.

If, on the other hand, the imaginary part additionally obeys $\partial_k^2 \omega_{k_c}|_{\underline{r}_c} = \partial_k^3 \omega_{k_c}|_{\underline{r}_c} = 0$ the second Kronecker-delta in (20) does not arise and there is no dynamics on the corresponding timescales. Hence there is no non-local contribution to the amplitude equation. Except for a fourth-order derivative the resulting amplitude equation resembles the usual Ginzburg–Landau equation in ordinary space. It should be mentioned that all these peculiarities disappear in the non-degenerate case.

Since the degenerate travelling wave instability is represented by a single point in the three-dimensional space of bifurcation parameters (cf figure 1) it will be difficult to attain in numerical simulations. In most cases one will be in some finite though small distance to it. It is therefore desirable to enlarge the validity of (21) to a neighbourhood of the codimension-three point. This will be achieved by discussing a rather general unfolding of the degenerate bifurcation. The corresponding calculations are deferred to appendix B. They lead to the result

$$\partial_{t_2} \psi_k^{(1)} = \left( \mu + \alpha_2 \left( \frac{k - k_c}{\sqrt{\varepsilon}} \right)^2 + \alpha_3 \left( \frac{k - k_c}{\sqrt{\varepsilon}} \right)^3 + \alpha_4 \left( \frac{k - k_c}{\sqrt{\varepsilon}} \right)^4 + \frac{2\gamma}{L} \sum_{k' \in \mathrm{I}_+} |\psi_{k'}^{(1)}|^2 \right) \psi_k^{(1)}. \tag{22}$$

The linear terms on the RHS of equation (22) contain additional quadratic and cubic contributions. The coefficients $\alpha_2$ and $\alpha_3$ are complex quantities which play the role of unfolding parameters. They vanish at the codimension-three bifurcation.

## 5. Solutions above threshold and their asymptotic behaviour

Due to the fact that the coupling terms in equation (22) affect only pairs of modes a rather complete discussion may be given as regards the existence of solutions, their asymptotic behaviour and their stability. It is sufficient to consider only real parameters $\mu, \alpha_i, \gamma$, because the time dependent transformation

$$\psi_k^{(1)} \to \psi_k^{(1)} \exp\left\{ -\mathrm{i} \left( \mathrm{Im}\,\{p(k)\} t_2 + 2\mathrm{Im}\,\{\gamma\} \int_0^{t_2} B(\tau)\mathrm{d}\tau \right) \right\}$$

where $p(k)$ and $B(t_2)$ are defined by

$$B(t_2) = \frac{1}{L} \sum_{k' \in \mathrm{I}_+} |\psi_{k'}^{(1)}|^2 \qquad p(k) = \mu + \alpha_2 \left( \frac{k - k_c}{\sqrt{\varepsilon}} \right)^2 + \alpha_3 \left( \frac{k - k_c}{\sqrt{\varepsilon}} \right)^3 + \alpha_4 \left( \frac{k - k_c}{\sqrt{\varepsilon}} \right)^4$$

removes all the imaginary parts of these coefficients. The polynomial $p(k)$ expresses the form of the spectrum in the interval of active modes and consequently shows one maximum, a saddle point or two maxima depending on the specific values of $\alpha_2$ and $\alpha_3$ (cf figure 1). Since the nonlinearity is essentially made up of the $k$-independent function $B(t_2)$, direct integration gives

$$\psi_k^{(1)}(t_2) = \psi_k^{(1)}(0) \cdot \mathrm{e}^{p(k)t_2} \cdot \mathrm{e}^{2\gamma \int_0^{t_2} B(\tau)\,\mathrm{d}\tau}. \tag{23}$$

$B$ has to be determined self-consistently from

$$-\frac{1}{4\gamma} \partial_{t_2} (\mathrm{e}^{-4\gamma \int_0^{t_2} B(\tau)\,\mathrm{d}\tau}) = \frac{1}{L} \sum_{k' \in \mathrm{I}_+} |\psi_{k'}^{(1)}(0)|^2 \mathrm{e}^{2p(k')t_2} =: g(t_2) \tag{24}$$

where the function $g$ is a known function of $t_2$ for any given initial condition $\psi_k^{(1)}(0)$. Integration of (24) yields immediately

$$\psi_k^{(1)}(t_2) = \psi_k^{(1)}(0) \cdot \mathrm{e}^{p(k)\,t_2} \cdot \left(1 - 4\gamma \int_0^{t_2} g(\tau)\,\mathrm{d}\tau\right)^{-\frac{1}{2}}. \tag{25}$$

We have thus obtained an analytical though somewhat formal solution of equation (22). It contains hitherto all sorts of solutions, stable and unstable as well as bounded and unbounded ones. In order to exclude unbounded solutions one has to demand that $\gamma < 0$. Furthermore, as all modes with $p(k) \leqslant 0$ will die out, attention has to be focused on the linearly unstable modes which are characterized by $p(k) > 0$.

We will prove in appendix C that any solution will run asymptotically—in fact irrespective of the initial conditions—into a definite single plane wave state. A rough argument explaining the emergence of such a simple stable space and time periodic final pattern is as follows: consider the squared Fourier modes†:

$$\begin{aligned} |\psi_k^{(1)}(t_2)|^2 &= |\psi_k^{(1)}(0)|^2 \cdot \mathrm{e}^{2p(k)t_2} \cdot \left(1 - 4\gamma \int_0^{t_2} g(\tau)\,\mathrm{d}\tau\right)^{-1} \\ &= \frac{|\psi_k^{(1)}(0)|^2 \cdot \mathrm{e}^{2p(k)t_2}}{1 - \frac{4\gamma}{L}\sum_{k'} \frac{|\psi_{k'}^{(1)}(0)|^2}{2p(k')}(\mathrm{e}^{2p(k')t_2} - 1)}. \end{aligned} \tag{26}$$

The most important contribution in the denominator stems from the fastest growing mode with wavenumber $k_M \in \mathrm{I}_+$. This need not, but can be $k_c$. Extracting it from the sum we write:

$$|\psi_k^{(1)}(t_2)|^2 = \frac{|\psi_k^{(1)}(0)|^2 \cdot \mathrm{e}^{2p(k_M)t_2}}{1 - \frac{4\gamma}{L}\frac{|\psi_{k_M}^{(1)}(0)|^2}{2p(k_M)}(\mathrm{e}^{2p(k')t_2} - 1) - \frac{4\gamma}{L}\sum_{k_M \neq k'} \frac{|\psi_{k'}^{(1)}(0)|^2}{2p(k')}(\mathrm{e}^{2p(k')t_2} - 1)}.$$

The denominator will cause all Fourier modes with $k \neq k_M$ to vanish as $t_2 \to \infty$. The only non-vanishing mode is the fastest growing one, in which case the numerator balances the denominator. Disregarding all the other modes, i.e. the sum for $t_2 \to \infty$ we are left with

$$|\psi_k^{(1)}(t_2)|^2 \to -\frac{p(k_M)}{2\gamma} L \delta_{k,k_M}. \tag{27}$$

The more careful analysis in appendix C leads to

$$|\psi^{(1)}(k, t_2)|^2 \to -\frac{p(k_M)}{\gamma} \pi \delta(k - k_M)$$

which is in full agreement with this conclusion.

To test these results numerically we perform a simple simulation of equation (22) for the ferromagnetic model which is presented in appendix A. The parameters $\gamma = 0.289\,17$, $\delta = -2.429\,12$, $\kappa = 1.117$ which are defined there, are chosen to be slightly above the codimension-three instability, which is located at $\gamma = 0.289\,19$, $\delta = -2.429\,23$, $\kappa = 1.117\,08$. $\Gamma = 0.1$, $j = 1$, $h_0 = 0.5$, $a = 2.247\,78$ are held fixed. The real part of the spectrum then takes the form shown in figure 3.

The critical wavenumber is $k_c = 1.166\,27$. Within certain limits the width of the interval $\mathrm{I}_+$ used in the derivation of the amplitude equation may be chosen arbitrarily. From the above picture one sees that $\Delta = 0.04$ is an apt choice. The Taylor-expansion of the spectrum yields the coefficients of $p(k)$: $\mu = 6.0 \times 10^{-10}$, $\alpha_2 = 5.0 \times 10^{-5}$, $\alpha_3 = 4.17 \times 10^{-3}$, $\alpha_4 = -4.331$. Note that the simulations make no use of the formal smallness parameter $\varepsilon$. The coefficient of the nonlinear term in (22) is given by (17) and turns out to be $2\gamma = -6.558 \times 10^{-2}$. The

† In these equations we assume $k' \in \mathrm{I}_+$, but omit the explicit notation at the sums.

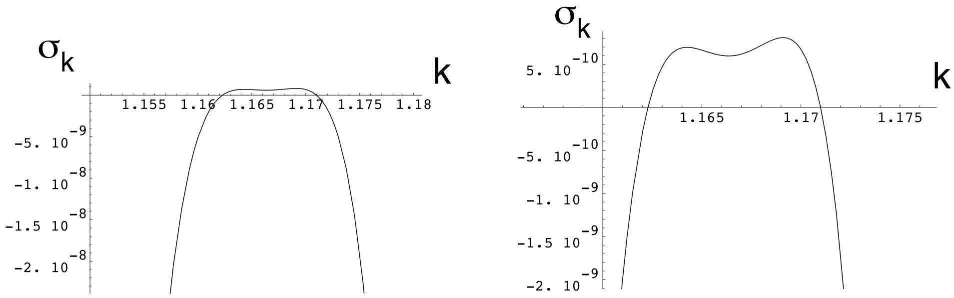


**Figure 3.** Real part of spectrum for $h_0 = 0.5$, $\Gamma = 0.1$, $j = 1$.

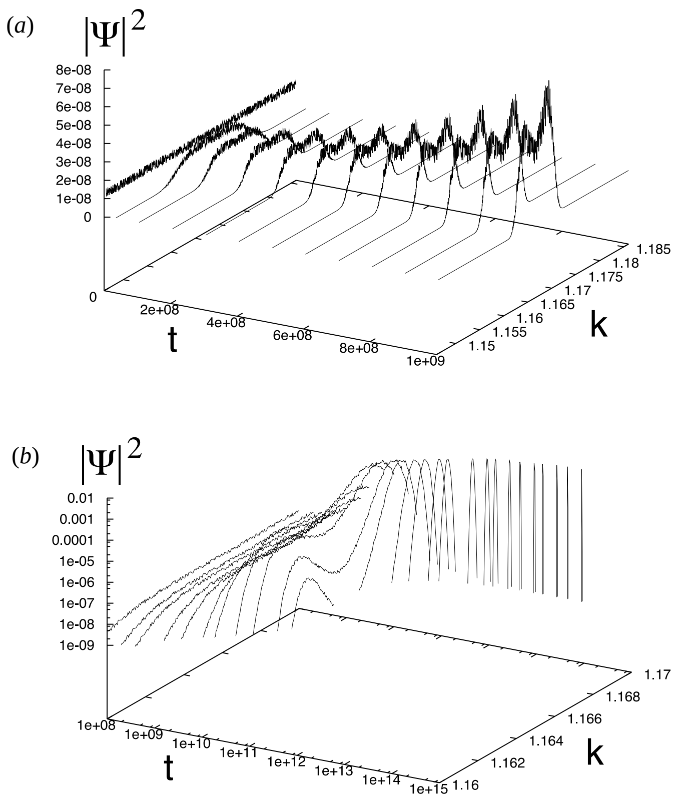


**Figure 4.** Simulations for $h_0 = 0.5$, $\Gamma = 0.1$, $j = 1$. (*a*) Shows the evolution of Fourier modes on a short timescale. (*b*) Exhibits the long time behaviour.

simulation uses 1024 wavenumbers in $I_+$ and a homogeneous initial condition on which white noise is superimposed.

Figure 4 shows that all modes with negative values of $\sigma_k$ die out very fast during the first integration steps. On the other hand the linearly unstable modes grow on an intermediate timescale in a way that is dominated by the form of the spectrum $\sigma_k$. On a larger scale, however, nearly all of these modes begin to decrease. Asymptotically only the linearly most unstable mode survives. The final peak is found at $k = 1.16\,913$. The square of the modulus of this mode is $2.0044 \times 10^{-3}$ which is in good agreement with the value $2.0046 \times 10^{-3}$ given by equation (27).

## 6. Summary

Motivated by a study of a damped and driven one-dimensional ferromagnet we were led to analyse a degenerate travelling wave instability in non-inversion symmetric one-dimensional systems quite generally. A weakly nonlinear analysis above threshold using a non-standard version of the method of multiple scales leads to amplitude equations for the spatial Fourier modes. Their striking feature is that there is only a nonlinear coupling between pairs of modes which are located symmetrically with respect to the origin. This fact simplifies the treatment of the equations considerably. A complete solution could be obtained. A discussion of its asymptotic behaviour showed that the most unstable wavenumbers dominate the dynamics of the system. In the end a long time equilibrium state being just a single plane wave is reached. This result agrees fully with numerical simulations.

## Acknowledgments

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## Appendix A. Ferromagnetic model system

On a macroscopic scale the equation of motion of the classical density of magnetization $\vec{M}(\vec{r}, t)$ of a ferromagnet is given by the Landau–Lifshitz equation.

$$\partial_t \vec{M}(\vec{r}, t) = -\vec{M}(\vec{r}, t) \times (\vec{H}_{eff(\vec{r},t)} + \Gamma\ \vec{M}(\vec{r}, t) \times \vec{H}_{eff}(\vec{r}, t)). \qquad \text{(A.1)}$$

The first term gives rise to a precession of $\vec{M}$ around the local effective magnetic field $\vec{H}_{eff}$, while the second—dissipative—term tends to align the local magnetic moment parallel to that field. This equation was proposed by Landau [9] phenomenologically, but was also derived from the underlying microscopic equations by Garanin and Plefka later on [10]. The modulus of the magnetization is a constant of the motion which may be set equal to 1.

In most cases the treatment of spatially three-dimensional phenomena is hopelessly complicated and beyond the scope of this article. We restrict ourselves to an investigation of an one-dimensional model. This case is realized either in ferromagnets with only one large aspect ratio coordinate or in systems where the magnetization is nearly homogeneous in a plane but varies in a direction perpendicular to it. In such a case the dipolar fields can be incorporated in an effective anisotropy. The effective magnetic field $\vec{H}_{eff}$ is then composed of external static and driving fields, as well as internal isotropic exchange and uniaxial anisotropy fields:

$$\vec{H}_{eff}(\vec{r}, t) = h_z\vec{e}_z + h_o(\cos\theta\,\vec{e}_x + \sin\theta\vec{e}_y) + j\nabla^2\vec{M} + aM_z\vec{e}_z \qquad \theta = \omega t + \kappa z. \qquad \text{(A.2)}$$

The driving field is a magnetic wave which is assumed to be circularly polarized and perpendicular to the static field. The running wave distinguishes a certain direction in the system, which is chosen to be the $z$-axis. Pattern formation along this direction is analysed. Spatially homogeneous driving fields were studied by Matthäus and Sauermann [5]. Performing the transformation $\vec{m} = R_z(\theta)\vec{M}$ where $R_z$ is the rotation matrix with respect to the $z$-axis the explicit dependence on space and time coordinates is eliminated and one obtains

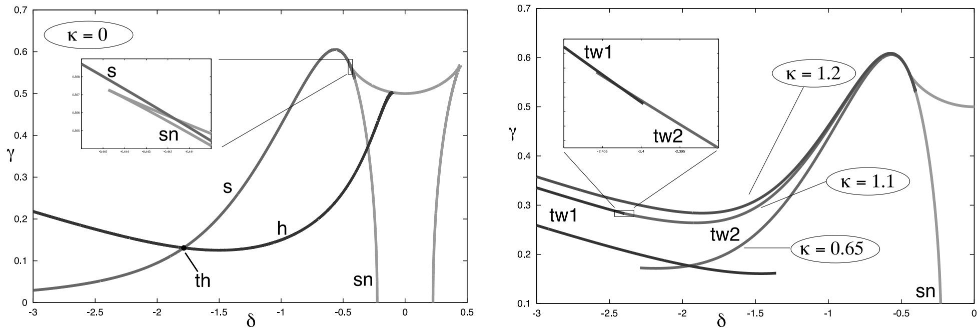


**Figure A1.** Bifurcation diagrams for $h_0 = 0.5$, $\Gamma = 0.1$, $j = a - \kappa^2 = 1$. s: soft mode, h: hard mode, tw: travelling wave, sn: saddle node line, th: Turing–Hopf point.

an autonomous partial differential equation for $\vec{m}$. Applying the stereographic projection, $\phi = \frac{m_x + \mathrm{i} m_y}{1+m_z}$ which takes care of the conservation of the modulus of $\vec{m}$ automatically the equation of motion becomes

$$\partial_t \phi = (\mathrm{i} - \Gamma)\Bigg[(\delta + \mathrm{i}\gamma)\phi - \frac{h_0}{2}(1-\phi^2) + j\left(\frac{2\phi^*(\partial_z\phi)^2}{1+|\phi|^2} - \partial_z^2\phi\right) + \left((a + j\kappa^2)\phi - 2\mathrm{i} j\kappa\partial_z\phi\right)\cdot\frac{1-|\phi|^2}{1+|\phi|^2}\Bigg]. \tag{A.3}$$

The parameters $\gamma = \Gamma\frac{\omega}{1+\Gamma^2}$ and $\delta = h_z - \frac{\omega}{1+\Gamma^2}$ denote a renormalized frequency and the detuning from resonance respectively. The fact that the inversion symmetry $z \to -z$ is broken for $\kappa \neq 0$ prevents the emergence of counterpropagating waves. Equation (A.3) permits either two or four stationary, spatially homogeneous solutions $\phi_0$. The corresponding regions in the $\gamma$–$\delta$-plane are separated by saddle node bifurcation lines. For the special case of a spatially homogeneous driving field, i.e. $\kappa = 0$, a situation which has been studied in detail in [5], there are other instabilities: soft- and hard-mode or Hopf bifurcations, respectively. They intersect at a Turing–Hopf point (cf figure A1). The box in the upper left corner makes evident that the soft-mode line does not meet the saddle-node line in the cusp point. In the upper part of this diagram the system has one stable and one unstable solution $\phi_0$. Inside the saddle-node lines it has four stationary homogeneous solutions. We note that these solutions are independent of $\kappa$. So the saddle-node lines are not affected by changing this parameter.

For finite $\kappa$ (cf figure A1), the soft- and hard-mode lines change their character: they become travelling wave instabilities whose frequency $\omega_c$ and wavenumber $k_c$ vary along the bifurcation lines. A new feature is that they have definite end points. These points approach each other with increasing $\kappa$ until they merge for $\kappa = 1.117\,08\ldots$ (cf figure A1 and its magnified insert). This is exactly the degenerated situation discussed in this article. Separating real and imaginary parts in (A.3) and subtracting $\phi_0$ yields an equation of motion of the general type (2). We add that the parameters $\rho_1$, $\rho_2$, $\rho_3$ of the schematic bifurcation diagram of section 1 correspond to $\delta$, $\gamma$, $\kappa$ in this example.

## Appendix B. Unfolding

In this section the concept of unfolding a bifurcation is applied to the degenerate travelling wave instability along the lines of [11]. The real part of the spectrum $\sigma_k^{(\nu_c)}$ of the linearized problem is proportional to $(k-k_c)^4$ at the codimension-three point. In the neighbourhood of this point the quartic maximum breaks up in two quadratic maxima. One or both of them may be positive, zero or negative (see figure 1 which illustrates several special cases). In order to understand this complicated situation comprehensively we admit infinitesimal but otherwise arbitrary perturbations in the parameters of the system. In terms of the smallness parameter $\varepsilon$ they are written as

$$\underline{r} = \underline{r}_c + \sqrt{\varepsilon}\underline{r}_{1/2} + \varepsilon\underline{r}_1 + \varepsilon\sqrt{\varepsilon}\underline{r}_{3/2} + \varepsilon^2\underline{r}_2 + \cdots . \tag{B.1}$$

Although the structure of equation (16) is not changed when applying the procedure outlined in section 3 the coefficients $\Omega_k^{(j)}$ acquire many additional terms. However, unlike in equation (17) the coefficients are not *a priori* real. Using the notation $r_l = |\underline{r}_l|$ and $r_l\partial_{e_l} = (\underline{r}_l \cdot \partial_{\underline{r}})$ where $\partial_{\underline{r}}$ denotes the gradient and $l \in \{\frac{1}{2}, 1, \frac{3}{2}, 2\}$ they can be written in the following way†:

$$\begin{aligned}
\Omega_k^{(1/2)} &= -\mathrm{i}r_{1/2}\partial_{\mathrm{e}_{1/2}}\lambda_c \qquad \hat{k} = (k - k_c)/\sqrt{\varepsilon} \\
\Omega_k^{(1)} &= \tfrac{1}{2}\hat{k}^2\partial_k^2\omega_c - \mathrm{i}\hat{k}r_{1/2}\partial_{\mathrm{e}_{1/2}}\partial_k\lambda_c - \mathrm{i}r_1\partial_{\mathrm{e}_1}\lambda_c - \frac{\mathrm{i}}{2}r_{1/2}^2\partial_{\mathrm{e}_{1/2}}^2\lambda_c \\
\Omega_k^{(3/2)} &= \tfrac{1}{6}\hat{k}^3\,\partial_k^3\lambda_c + \tfrac{1}{2}\hat{k}^2r_{1/2}\partial_{\mathrm{e}_{1/2}}\partial_k^2\lambda_c + \tfrac{1}{2}\hat{k}r_{1/2}^2\partial_{\mathrm{e}_{1/2}}^2\partial_q\lambda_c \\
&\quad + \hat{k}r_1\partial_{\mathrm{e}_1}\partial_q\lambda_c + \tfrac{1}{6}r_{1/2}^3\partial_{\mathrm{e}_{1/2}}^3\lambda_c + r_1r_{1/2}\partial_{\mathrm{e}_1}\partial_{\mathrm{e}_{1/2}}\lambda_c + r_{3/2}\partial_{\mathrm{e}_{3/2}}\lambda_c \\
\mu + \alpha_k &= \frac{1}{4!}\hat{k}^4\partial_q^4\lambda_c + \tfrac{1}{6}\hat{k}^3r_{1/2}\partial_{e_{1/2}}\partial_q^3\lambda_c + \tfrac{1}{4}\hat{k}^2r_{1/2}^2\partial_{e_{1/2}}^2\partial_q^2\lambda_c + \tfrac{1}{2}\hat{k}^2r_1\partial_{e_1}\partial_q^2\lambda_c \\
&\quad + \tfrac{1}{6}\hat{k}r_{1/2}^3\partial_{e_{1/2}}^3\partial_q\lambda_c + \hat{k}r_1r_{1/2}\partial_{e_1}\partial_{e_{1/2}}\partial_q\lambda_c + \hat{k}r_{3/2}\,\partial_{e_{3/2}}\partial_q\lambda_c + \frac{1}{4!}r_{1/2}^4\partial_{e_{1/2}}^4\lambda_c \\
&\quad + \tfrac{1}{2}r_{1/2}^2\partial_{e_1}\partial_{e_{1/2}}^2\lambda_c + \tfrac{1}{2}r_1^2\partial_{e_1}^2\lambda_c + r_{3/2}r_{1/2}\partial_{e_{3/2}}\partial_{e_{1/2}}\lambda_c + r_2\partial_{e_2}\lambda_c .
\end{aligned} \tag{B.2}$$

One establishes that the perturbation theoretical treatment described after equation (17) remains basically unchanged. The differences that arise stem from the fact that the $\Omega_k^{(j)}$ are now complex quantities. As a consequence certain conditions will have to be fulfilled in order that the solutions of equations (16) retain their oscillatory behaviour. Otherwise the solutions would die out or increase exponentially in time. These conditions lead to restrictions on the possible directions of the vectors $\underline{r}_l$. First of all the vector $\underline{r}_{1/2}$ must be tangent to the bifurcation surface, since the real part of the linearized spectrum is zero anywhere on the bifurcation surface. Consequently $(r_{1/2}\partial_{e_{1/2}})^n\lambda_c$ is purely imaginary for any $n$ and so

$$\psi_k^{(1)} = \zeta_k^{(1)}\mathrm{e}^{\mathrm{i}\Omega_k^{(1/2)}t_{1/2}} \qquad \psi_k^{(3/2)} = \psi_k^{(3/2)}(t_{1/2}, t_1, \ldots). \tag{B.3}$$

By the same token the terms $(r_{1/2}\partial_{e_{1/2}})^n\partial_k\lambda_c$ are also purely imaginary since the instability conditions also require the real part of $\partial_k\lambda_c$ to be zero on that surface.

Therefore all terms in $\Omega_k^{(1)}$ except $r_1\partial_{e_1}\lambda_c$ are real. To avoid imaginary parts altogether $\underline{r}_1$ also has to be tangent to the bifurcation surface. $\underline{r}_1$ and $\underline{r}_{1/2}$ are linearly independent vectors which may be chosen to be perpendicular to each other. This gives up to order $m = 4$.

$$\psi_k^{(2)} = \psi_k^{(2)}(t_{1/2}, t_1, \ldots) \qquad \psi_k^{(3/2)} = \zeta_k^{(3/2)}\mathrm{e}^{\mathrm{i}\Omega_k^{(1/2)}t_{1/2}} \qquad \zeta_k^{(1)} = \eta_k^{(1)}\mathrm{e}^{\mathrm{i}\Omega_k^{(1)}t_1}. \tag{B.4}$$

Analogous considerations apply for $m = 5$. It turns out that $\underline{r}_{3/2}$ must also lie in the bifurcation surface. But since the tangent plane of the bifurcation surface is two-dimensional it may be written as a linear combination of $\underline{r}_1$ and $\underline{r}_{1/2}$ and thus does not produce independent new contributions. In the same order an additional condition on the direction of $\underline{r}_{1/2}$ arises. It must be a tangent vector to another surface, which is defined by $\partial_k^2\sigma_k|_c = 0$. We therefore conclude that the direction of $\underline{r}_{1/2}$ is fixed completely. Note that $\partial_{e_{1/2}}\partial_{e_1}\lambda$ is purely imaginary

† The subscript $c$ in $\square_c$ replaces the lengthier form $\square_k^{(\nu_c)}|_{k_c,\underline{r}_c}$ in these equations.

since both $\underline{r}_{1/2}$ and $\underline{r}_1$ are tangent to the bifurcation surface. As in equation (20) only terms with $k^2 = k_1^2 + k_2^2 - k_3^2$ enter the secular condition. Therefore,

$$\begin{aligned} \psi_k^{(5/2)} &= \psi_k^{(5/2)}(t_{1/2}, t_1, \ldots) \qquad \psi_k^{(2)} = \zeta_k^{(2)} \mathrm{e}^{\mathrm{i}\Omega_k^{(1/2)} t_{1/2}} \\ \zeta_k^{(3/2)} &= \eta_k^{(3/2)} \mathrm{e}^{\mathrm{i}\Omega_k^{(1)} t_1} \qquad \eta_k^{(1)} = \xi_k^{(1)} \mathrm{e}^{\mathrm{i}\Omega_k^{(3/2)}(k) t_{3/2}}. \end{aligned} \tag{B.5}$$

With these choices of $\underline{r}_{1/2}, \underline{r}_1, \underline{r}_{3/2}$, all frequencies $\Omega_k^{(j)}$ are real. Following the line of reasoning that led from (20) to (21) one gets an equation for the dynamics of $\zeta_k^{(1)}$ whose nonlinear part contains only contributions of pairs of modes:

$$(\partial_{t_2} - \mathrm{i}\alpha_1 \hat{k})\xi_k^{(1)} = \mu\xi_k^{(1)} + \alpha_2 \hat{k}^2 \xi_k^{(1)} + \alpha_3 \hat{k}^3 \xi_k^{(1)} + \alpha_4 \hat{k}^4 \xi_k^{(1)} + \frac{2\gamma}{L}\left(\sum_{k' \in \mathrm{I}_+} |\xi_k k'|^2\right)\xi_k^{(1)}. \tag{B.6}$$

The coefficients are defined by

$$\begin{aligned} \mu &= \frac{1}{4!} r_{1/2}^4 \partial_{e_{1/2}}^4 \lambda_c + \tfrac{1}{2} r_1 r_{1/2}^2 \partial_{e_1} \partial_{e_{1/2}}^2 \lambda_c + \tfrac{1}{2} r_1^2 \partial_{e_1}^2 \lambda_c + r_{3/2} r_1 \partial_{e_{3/2}} \partial_{e_1} \lambda_c + r_2 \partial_{e_2} \lambda_c \\ \alpha_1 &= \tfrac{1}{6} r_{1/2}^3 \partial_{e_{1/2}}^3 \partial_k \omega_c + r_1 r_{1/2} \partial_{e_1} \partial_{e_{1/2}} \partial_k \omega_c + r_{3/2} \partial_{e_{3/2}} \partial_k \omega_c \qquad \alpha_4 = \frac{1}{4!} \partial_k^4 \lambda_c \\ \alpha_2 &= \tfrac{1}{4} r_{1/2}^2 \partial_{e_{1/2}}^2 \partial_k^2 \lambda_c + \tfrac{1}{2} r_1 \partial_{e_1} \partial_k^2 \lambda_c \qquad \alpha_3 = \tfrac{1}{6} r_{1/2} \partial_{e_{1/2}} \partial_k^3 \lambda_c. \end{aligned} \tag{B.7}$$

Obviously a Galilei-transformation helps to get rid of $\alpha_1$. For $r_{1/2} = r_1 = r_{3/2} = 0$ the coefficients $\alpha_1, \alpha_2, \alpha_3$ vanish. The unfolded version of (21) is thus:

$$\partial_{t_2} \psi_k^{(1)} = \left(\mu + \alpha_2 \left(\frac{k - k_c}{\sqrt{\varepsilon}}\right)^2 + \alpha_3 \left(\frac{k - k_c}{\sqrt{\varepsilon}}\right)^3 + \alpha_4 \left(\frac{k - k_c}{\sqrt{\varepsilon}}\right)^4 + \frac{2\gamma}{L} \sum_{k' \in \mathrm{I}_+} |\psi_{k'}^{(1)}|^2\right)\psi_k^{(1)}. \tag{B.8}$$

## Appendix C. Asymptotic expansion of $g(\tau)$ for $L \to \infty$

It is evident from equation (26), that the asymptotic values of the $\psi_k^{(1)}$ will be determined by a balancing argument which decides whether the numerator, i.e. $\exp(2p(k)t_2)$ or the denominator, i.e. $\int_0^{t_2} g(\tau)\,\mathrm{d}\tau$ grows faster. For $L \to \infty$ the sum over $k$ in the definition of $g(t)$, equation (24) is replaced by an integral. Carrying out the time integration one finds

$$\begin{aligned} \int_0^{t_2} g(\tau)\,\mathrm{d}\tau &= \int_0^{t_2} \frac{1}{L} \sum_{k \in \mathrm{I}_+} |\psi_k^{(1)}(0)|^2 \mathrm{e}^{2p(k)\tau}\,\mathrm{d}\tau \\ &= \frac{1}{2\pi} \int_{\mathrm{I}_+} \frac{|\psi^{(1)}(k,0)|^2}{2p(k)} (\mathrm{e}^{2p(k)t_2} - 1)\,\mathrm{d}k. \end{aligned} \tag{C.1}$$

For $t_2 \to \infty$ only the unstable modes survive. Neglecting the summand $-1$ and assuming that $\alpha_3 \neq 0$, so that $p(k)$ has only a single maximum† $k_M$, we expand $p(k)$ in a Taylor series around $k_M$. We arrive at

$$\int_0^{t_2} g(\tau)\,\mathrm{d}\tau = \frac{\mathrm{e}^{2p(k_M)t_2}}{2\pi} \int_{-c_u}^{c_o} \frac{|\psi^{(1)}(k_M + \xi/\sqrt{\beta_2 t_2}, 0)|^2}{2p(k_M + \xi/\sqrt{\beta_2 t_2})} \mathrm{e}^{\beta_3 \frac{\xi^3}{\sqrt{t_2}} - \beta_4 \frac{\xi^4}{t_2}} \mathrm{e}^{-\xi^2} \frac{\mathrm{d}\xi}{\sqrt{\beta_2 t_2}} \tag{C.2}$$

where we have introduced a new variable $\xi = (k - k_M)\sqrt{\beta_2 t_2}$ as well as the abbreviations

$$\begin{aligned} c_u &= (k_M - k_c + \Delta/2)\sqrt{\beta_2 t_2} \qquad c_o = (k_c - k_M + \Delta/2)\sqrt{\beta_2 t_2} \\ \beta_2 &= |\partial_k^2 p(k_M)| \qquad \beta_3 = \frac{\partial_k^3 p(k_M)}{3\sqrt{\beta_2}^3} \qquad \beta_4 = \frac{\partial_k^4 p(k_M)}{12\beta_2^2}. \end{aligned} \tag{C.3}$$

† The case $\alpha_3 = 0$ is similar, but one has to deal with the two most unstable modes. This only produces more terms to consider but all arguments remain essentially unchanged.

For $t_2 \to \infty$ this integral whose integrand is the product of a very slowly varying function with a Gaussian may be extended from $-\infty$ to $\infty$. Now the asymptotic expansion of the integral is performed by expanding the slowly varying part in a Taylor series at $\xi = 0$ and an integration by parts:

$$\int_0^{t_2} g(\tau)\,\mathrm{d}\tau = \frac{1}{2\pi}\frac{\mathrm{e}^{2p(k_M)t_2}}{\sqrt{\beta_2 t_2}}\left(\frac{|\psi^{(1)}(k_M,0)|^2\sqrt{\pi}}{2p(k_M)} + \mathrm{O}(1/\sqrt{t_2})\right). \tag{C.4}$$

Inserting this result into equation (26) yields

$$\begin{aligned}|\psi^{(1)}(k,t_2)|^2 &= |\psi^{(1)}(k,0)|^2\mathrm{e}^{2p(k)t_2}\left(-4\gamma\frac{1}{2\pi}\frac{\mathrm{e}^{2p(k_M)t_2}}{\sqrt{\beta_2 t_2}}\frac{|\psi^{(1)}(k_M,0)|^2\sqrt{\pi}}{2p(k_M)}\right)^{-1}\\ &= -\frac{p(k_M)\pi}{\gamma}\frac{|\psi^{(1)}(k,0)|^2}{|\psi^{(1)}(k_M,0)|^2}\sqrt{\frac{\beta_2 t_2}{\pi}}\mathrm{e}^{2(p(k)-p(k_M))t_2}\end{aligned} \tag{C.5}$$

where again the summand $-1$ as well as terms of $\mathrm{O}(1/\sqrt{t})$ were neglected. As the two last factors tend to the Dirac delta function, we get the result quoted in section 4.

$$|\psi^{(1)}(k,t_2\to\infty)|^2 = -\frac{p(k_M)}{\gamma}\pi\delta(k-k_M). \tag{C.6}$$

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